

Distributivity and the greatest common divisor

It is known that multiplication by a natural number distributes over the greatest common divisor. It is an important property that can be used to simplify arguments where both multiplication and the greatest common divisor are involved. A nice example is Netty van Gasteren's proof (see [Dij01]) of the theorem

$$(0) \quad [(m \times p) \nabla n = m \nabla n \iff p \nabla n = 1] ,$$

where variables m , n , and p are of type integer and ∇ is a natural-valued operator that stands for the greatest common divisor. We read $m \nabla n$ as "m nabra n" and the square brackets denote universal quantification over all free variables. Netty's proof is as follows:

$$\begin{aligned} & m \nabla n \\ = & \quad \{ \quad p \nabla n = 1 \text{ and } 1 \text{ is the unit of multiplication} \quad \} \\ & (m \times (p \nabla n)) \nabla n \\ = & \quad \{ \quad \text{multiplication by a natural number distributes over } \nabla \\ & \quad \text{and } \nabla \text{ is associative} \quad \} \\ & (m \times p) \nabla (m \times n) \nabla n \\ = & \quad \{ \quad (m \times n) \nabla n = n \quad \} \\ & (m \times p) \nabla n . \end{aligned}$$

Remark Although m is an integer, we can safely apply the distributivity property in the second step, because we can freely change the sign of the arguments of ∇ — that is, $[(-m) \nabla n = m \nabla n]$. (**End of Remark**)

Similar to the multiplication by a natural number, there are other functions that distribute over ∇ . The goal of this note is to determine reasonable sufficient conditions for a natural-valued function f to distribute over ∇ , i.e., for the following property to hold:

$$(1) \quad [f.(m \nabla n) = f.m \nabla f.n] .$$

For simplicity's sake, we restrict all variables to naturals and extend the results to integers later. This implies that the domain of f is also restricted to the natural numbers.

Due to our general line of research, we want to prove (1) by exploiting invariants of Euclid's algorithm involving the function f . Recall that Euclid's algorithm can be written (for positive arguments m and n) as:

$$\begin{array}{l}
 \{ 0 < m \ \wedge \ 0 < n \} \\
 x, y := m, n; \\
 \{ \text{Invariant: } 0 < x \ \wedge \ 0 < y \ \wedge \ m \nabla n = x \nabla y \} \\
 \text{do } y < x \rightarrow x := x - y \\
 \square \ x < y \rightarrow y := y - x \\
 \text{od} \\
 \{ 0 < x \ \wedge \ 0 < y \ \wedge \ x = m \nabla n \ \wedge \ y = m \nabla n \}
 \end{array}$$

To determine an appropriate loop invariant, we take the right-hand side of (1) and we observe:

$$\begin{array}{l}
 f.m \nabla f.n \\
 = \quad \{ \text{initially: } x = m \ \wedge \ y = n \} \\
 f.x \nabla f.y \\
 = \quad \{ \text{suppose that } f.x \nabla f.y \text{ is invariant;} \\
 \quad \text{on termination: } x = m \nabla n \ \wedge \ y = m \nabla n \} \\
 f.(m \nabla n) \nabla f.(m \nabla n) \\
 = \quad \{ \nabla \text{ is idempotent} \} \\
 f.(m \nabla n) .
 \end{array}$$

Property (1) is thus established under the assumption that $f.x \nabla f.y$ is an invariant of the loop body.

Remark “Invariants” in the literature are always boolean-valued functions of the program variables. But we see no reason why “invariants” shouldn't be of any type: for us, an *invariant* of a loop is simply a function of the program variables whose value is unchanged by execution of the loop body. In this case, the value is a natural number. (End of Remark)

The next step is to determine what condition on f guarantees that $f.x \nabla f.y$ is indeed invariant. Noting the symmetry in the loop body between x and y , the condition is

easily calculated to be

$$\left[f.(x - y) \nabla f.y = f.x \nabla f.y \Leftrightarrow 0 < y < x \right] .$$

Equivalently, by the rule of range translation ($x := x + y$), the condition can be written as

$$(2) \left[f.x \nabla f.y = f.(x + y) \nabla f.y \Leftrightarrow 0 < x \wedge 0 < y \right] .$$

Formally, this means that

$$\text{“ } f \text{ distributes over } \nabla \text{”} \Leftrightarrow (2) .$$

Incidentally, the converse of this property is also valid:

$$(2) \Leftrightarrow \text{“ } f \text{ distributes over } \nabla \text{”} .$$

To prove it, we use the theorem

$$(3) \left[(m + a \times n) \nabla n = m \nabla n \right] ,$$

and we calculate:

$$\begin{aligned} & f.(x + y) \nabla f.y \\ = & \left\{ \text{ } f \text{ distributes over } \nabla \right\} \\ & f.((x + y) \nabla y) \\ = & \left\{ (3) \right\} \\ & f.(x \nabla y) \\ = & \left\{ \text{ } f \text{ distributes over } \nabla \right\} \\ & f.x \nabla f.y . \end{aligned}$$

From the mutual implication we conclude that

$$\text{“ } f \text{ distributes over } \nabla \text{”} \equiv (2) .$$

We have now reached a point where we can determine if a function distributes over ∇ . However, since (2) still has two occurrences of ∇ , we want to refine it into simpler properties. Towards that end we turn our attentions to the condition

$$f.x \nabla f.y = f.(x + y) \nabla f.y ,$$

and we try to calculate one side of it to the other. For instance, using theorem (3), it is immediate that any function that distributes over addition distributes over ∇ (note that this is the case of multiplication by a natural number). The proof is very simple:

$$\begin{aligned}
& f.(x + y) \nabla f.y \\
= & \{ \quad f \text{ distributes over addition} \quad \} \\
& (f.x + f.y) \nabla f.y \\
= & \{ \quad (3) \quad \} \\
& f.x \nabla f.y .
\end{aligned}$$

In view of properties (3) and (0), we formulate the following lemma, which is a more general requirement:

Lemma 4 All functions f that satisfy

$$\langle \forall x, y :: \langle \exists a, b : a \nabla f.y = 1 : f.(x + y) = a \times f.x + b \times f.y \rangle \rangle$$

distribute over ∇ .

Proof

$$\begin{aligned}
& f.(x + y) \nabla f.y \\
= & \{ \quad f.(x + y) = a \times f.x + b \times f.y \quad \} \\
& (a \times f.x + b \times f.y) \nabla f.y \\
= & \{ \quad (3) \quad \} \\
& (a \times f.x) \nabla f.y \\
= & \{ \quad a \nabla f.y = 1 \quad \text{and} \quad (0) \quad \} \\
& f.x \nabla f.y .
\end{aligned}$$

Please note that since the discussion above is based on Euclid's algorithm, it only applies to positive arguments. We now investigate the case where m or n are 0. We have, for $m = 0$:

$$\begin{aligned}
& f.(0 \nabla n) = f.0 \nabla f.n \\
= & \{ \quad [0 \nabla m = m] \quad \} \\
& f.n = f.0 \nabla f.n \\
= & \{ \quad f.n \text{ is a divisor of } f.0 \quad \} \\
& f.n \setminus f.0 \\
= & \{ \quad \text{definition} \quad \}
\end{aligned}$$

$$\begin{aligned} & \langle \exists k : k \in \mathbb{Z} : f.0 = k \times f.n \rangle \\ \Leftrightarrow & \{ \text{obvious possibilities for } f.0 \text{ or for } f.n \} \\ & f.0 = 0 \vee f.n = 1 \vee f.n = f.0 . \end{aligned}$$

Hence, using the symmetry between m and n we have, for $m = 0$ or $n = 0$:

$$(5) \quad f.(m \nabla n) = f.m \nabla f.n \quad \Leftrightarrow \quad f.0 = 0 \vee f.n = 1 \vee f.n = f.0 .$$

The conclusion is that we can use (5) and Lemma 4 to prove that a natural-valued function with domain \mathbb{N} distributes over ∇ .

Example 0: the Fibonacci function

In [Dij90], Edsger Dijkstra proves that the Fibonacci function distributes over ∇ . He does not use Lemma 4 explicitly, but he constructs the property

$$(6) \quad \text{fib.}(x + y) = \text{fib.}(y - 1) \times f.x + \text{fib.}(x + 1) \times \text{fib.}y ,$$

and then, using the lemma

$$\text{fib.}y \nabla \text{fib.}(y - 1) = 1 ,$$

he concludes the proof. His calculation is the same as that in the proof of Lemma 4 but for particular values of a and b and with f replaced by fib . Incidentally, if we don't want to construct property (6) we can easily verify it using induction — more details are given in [GKP94].

Example 1: the Mersenne function

In this subsection we prove that, for all integers k and m such that $k^m > 0$, the function defined as

$$f.m = k^m - 1$$

distributes over ∇ .

First, we observe that $f.0 = 0$. Next, we use Lemma 4. This means that we need to find integers a and b , such that

$$k^{m+n} - 1 = a \times (k^m - 1) + b \times (k^n - 1) \quad \wedge \quad a \nabla (k^n - 1) = 1 .$$

The most obvious instantiations for a are 1, k^n and $k^n - 2$. (That two consecutive numbers are coprime follows from (3).) Choosing $a = 1$, we calculate b :

$$\begin{aligned}
k^{m+n} - 1 &= (k^m - 1) + b \times (k^n - 1) \\
&= \{ \text{arithmetic} \} \\
k^{m+n} - k^m &= b \times (k^n - 1) \\
&= \{ \text{multiplication distributes over addition} \} \\
k^m \times (k^n - 1) &= b \times (k^n - 1) \\
\Leftarrow \{ \text{Leibniz} \} \\
k^m &= b .
\end{aligned}$$

We thus have

$$k^{m+n} - 1 = 1 \times (k^m - 1) + k^m \times (k^n - 1) \wedge 1 \nabla (k^n - 1) = 1 ,$$

and we use Lemma 4 to conclude that f distributes over ∇ :

$$[(k^m - 1) \nabla (k^n - 1) = k^{(m \nabla n)} - 1] .$$

In result, the Mersenne function, which is defined as $2^m - 1$, distributes over ∇ :

$$(7) \quad [(2^m - 1) \nabla (2^n - 1) = 2^{(m \nabla n)} - 1] .$$

A corollary of (7) is the property

$$[(2^m - 1) \nabla (2^n - 1) = 1 \quad \equiv \quad m \nabla n = 1] .$$

In words, two numbers $2^m - 1$ and $2^n - 1$ are coprime is the same as exponents m and n are coprime.

Extending the results to integers

A question that arises is whether we can extend the domain of the function f to the integer domain. To answer it, let us investigate when (1) holds for integer values. Assuming that m and n are integers, we calculate:

$$\begin{aligned}
& [f.(m \nabla n) = f.m \nabla f.n] \\
&= \{ \quad [m \nabla n = |m| \nabla |n|] \quad \} \\
& [f.(|m| \nabla |n|) = f.m \nabla f.n] \\
&= \{ \quad f \text{ distributes over } \nabla \text{ in the naturals} \quad \} \\
& [f.|m| \nabla f.|n| = f.m \nabla f.n] \\
\Leftarrow & \{ \quad \text{Leibniz} \quad \} \\
& [f.|m| = f.m] .
\end{aligned}$$

We thus conclude that:

$$\begin{aligned} & \text{“ } f \text{ distributes over } \nabla \text{ in the integers”} \\ \Leftrightarrow & \text{“ } f \text{ distributes over } \nabla \text{ in the naturals”} \wedge [f.\mid m = f.m] . \end{aligned}$$

Lemma 4 can be used to check if f distributes over ∇ in the naturals.

Acknowledgements

I thank Roland Backhouse, Alexandra Mendes, Arjan Mooij and Jeremy Weissmann for their valuable comments and help presenting parts of the note.

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May 8, 2007

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